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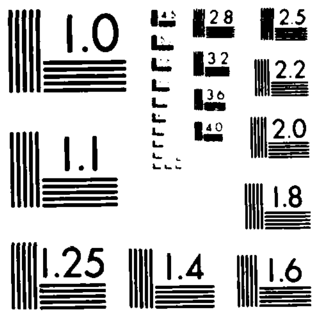

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$$\int^{\infty} e^{-\alpha s} F(m+|u|)ds > V(x)$$

WIENER-POISSON CONTROL PROBLEMS

BY

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1. Introduction. Let $W(t)$, $t \geq 0$, $W(0) = 0$ be a standard Wiener process, independent of $N(t)$, $t \geq 0$, $N(0) = 0$, a Poisson process with (constant) unit jumps, and $EN(t) = \lambda t$, $t \geq 0$. Let their sigma fields be $F(t) = \sigma(W(s), 0 \leq s \leq t)$ and $G(t) = \sigma(N(s), 0 \leq s \leq t)$, respectively. Let $X(t)$ be a stochastic process that (for $\frac{\partial X}{\partial t} \equiv X_t$) satisfies the Ito stochastic differential equation

$$(1.1) \qquad \begin{aligned} X_t(t) &= u(X(t))dt + dW(t) + dN(t) \\ X(0) &= x\ , \end{aligned}$$

where x is real, and $u(X(t))$ is measurable with respect to $\sigma(F(t) \cup G(t))$ (i.e. u is non-anticipative) and satisfies, for A, B constants, $B > 0$, and $|A| < B$,

$$(1.2) \qquad |u-A| \leq B$$

for all $0 \leq t \leq T$, $0 < T \leq \infty$ a constant. One cost function for a given u satisfying (1.2) is, for $\alpha > 0$ a constant, and $\varphi(x)$ a symmetric positive, increasing on the positive x-axis function of polynomial growth as $x \to \infty$, that is, for some $\beta > 0$,

$$(1.3) \qquad \varphi(x)/|x|^{\beta} \to 0 \text{ as } |x| \to \infty,\ \varphi(x) \to \infty, \text{ and } \varphi_{xx}(x) \to \infty \text{ as } x \to \infty.$$

$$(1.4) \qquad J(u) = \int_0^T e^{-\alpha s} E(\varphi(x(s)) + |u(X(s)|)ds.$$

Another cost function to be considered is

$$(1.5) \qquad K(u) = \int_0^T e^{-\alpha s} E(\varphi(X(s)) + u^2(X(s)))ds.$$

This latter cost function will be briefly treated in section 4.

The object is to characterize the optimal u for which J or K is minimized, respectively. The cases $T < \infty$ and $T = \infty$ are treated separately. The existence of an optimal u depends on assumptions about the asymptotic behavior of certain partial differential-difference equations. The method employs a suitable Bellman equation, a maximum principle for parabolic partial differential-difference equations and the Ito rule. The method follows [4].

2. Finite Interval Control.

Let $T < \infty$. Define, for $0 \leq t \leq T$,

$$(2.1) \qquad V = V(x,t) = \inf_{|u-A|<B} \int_0^t e^{-\alpha s} E(\varphi(X^2(s)) + |u(X(s))|)ds$$

and $\qquad V(x,0) = x$.

Writing $\int_0^t \equiv \int_0^h + \int_h^{t+h} - \int_t^{t+h}$, heuristic arguments (or see [2], pp. 179-180) yield a Bellman equation (where $V \equiv V(x,t)$, $V_x = \frac{\partial V}{\partial x}$, $V_{xx} = \frac{\partial^2 V}{\partial x^2}$, $u \equiv u(x)$)

$$(2.2) \qquad \varphi(x) + \inf_{|u-A|<B} (uV_x + |u|) + \frac{1}{2} V_{xx} - \alpha V - V_t$$

$$+ \lambda(V(x+1,t) - V) = 0.$$

Define

$$(2.3) \qquad g(a) \equiv \inf_{|u-A| \leq B} (ua+|u|) = \begin{cases} (B-A)(1-a) & \text{if } a \geq 1 \\ 0 & \text{if } |a| < 1 \\ (A+B)(1+a) & \text{if } a \leq -1 \end{cases}$$

Then (2.2), (2.3) become

$$(2.4) \qquad \varphi + g(V_x) + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t)-V) = 0.$$

On heuristic grounds, a solution to (2.4) is sought such that for functions $b_1(t) < b_2(t)$; $0 \leq t \leq T$ to be determined,

$$(2.5a) \qquad \varphi + (A+B)(1+V_x) + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t)-V) = 0$$

$$\text{for} \qquad x \leq b_1(t),$$

$$(2.5b) \qquad \varphi + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t)-V) = 0$$

$$\text{for} \qquad b_1(t) < x < b_2(t),$$

and

$$(2.5c) \qquad \varphi + (B-A)(1-V_x) + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t)-V) = 0$$

$$\text{for} \qquad x \geq b_2(t).$$

The functions $b_1(t)$, $b_2(t)$ are to be obtained from these matching conditions, where for $0 \leq t \leq T$,

$V \equiv V_1$ in (2.5a), $V \equiv V_2$ in (2.5b), $V \equiv V_3$ in (2.5c):

$$(2.6) \qquad \begin{aligned} V_1(b_1(t),t) &= V_2(b_1(t),t) \\ V_2(b_2(t),t) &= V_3(b_2(t),t) \end{aligned}$$

$$V_{1,x}(b_1(t),t) = V_{2,x}(b_1(t),t) = -1$$

$$V_{2,x}(b_2(t),t) = V_{3,x}(b_2(t),t) = +1.$$

(2.7) $V_1(x,0) = V_2(x,0) = V_3(x,0) = 0.$

(2.7a) $V_{3,xx}(b_2(t),t) \geq 0$

(2.7b) $V_{2,xx}(b_1(t),t) \geq 0.$

For R a constant, denote

(2.8) $J(x,t,R) \equiv \int_0^t e^{-\alpha s}\left[E(\varphi(Rs+W(s)+N(s)+x)+|R|\right]ds.$

It may be verified that $J(x,t,A+B)$ is a particular solution to (2.5a), that $J(x,t,0)$ is a particular solution to (2.5b) and that $J(x,t,A-B)$ is a particular solution to (2.5c). The solutions to (2.5a-c) will be shown to follow if this condition holds.

Assumption 1. There is a non-zero solution, for each t,

$$H_1(x,t) \text{ with } H_1(x,0) = 0 \text{ to } (H \equiv H(x,t))$$

(2.9) $(A+B)(1+H_x) + \frac{1}{2}H_{xx} - \alpha H - H_t + \lambda(H(x+1,t)-H) = 0$

such that

(2.10) $H_1(x,t) = O(e^{+rx})$, $H_{1,xx}(x,t) = O(e^{+\ell x})$

for some $r > 0$, $\ell > 0$, as $x \to -\infty$.

Also, there is a non-zero solution $H_2(x,t)$ with $H_2(x,0) = 0$ to

(2.11) $\frac{1}{2}H_{xx} - \alpha H - H_t + \lambda(H(x+1,t)-H) = 0.$

Further, there is a non-zero solution $H_3(x,t)$ with $H_3(x,0) = 0$ to

(2.12) $(3-A)(1-H_x) + \frac{1}{2}H_{xx} - \alpha H - H_t + \lambda(H(x+1,t)-H) = 0$

such that

$$(2.13)\qquad H_3(x,t) = O(e^{-kx})$$

$$H_{3,xx}(x,t) = O(e^{-cx})$$

for some $k > 0$, $c > 0$, all t, as $x \to +\infty$.

If the above Assumption 1 holds then let

$$(2.14a)\quad V_1(x,t) \equiv J(x,t,A+B) + H_1(x,t)$$

$$(2.14b)\quad V_2(x,t) \equiv J(x,t,0) + H_2(x,t)$$

$$(2.14c)\quad V_3(x,t) \equiv J(x,t,A-B) + H_3(x,t).$$

<u>Assumption 2.</u> The $V_\ell(x,t)$, $1 \le \ell \le 3$ of (2.14a-c) which satisfy (2.5a-c) respectively, and conditions (2.6), (2.7), *determine* $b_1(t) < b_2(t)$.

This motivates

<u>Theorem 1</u>. If the conditions of section 2 and Assumptions 1 and 2 hold for $0 \le t \le T < \infty$, then the optimal u_0 may be expressed in closed loop form as

$$(2.15)\qquad u_0(X_0(t)) = \begin{cases} A+B & \text{if} \quad X_0(t) \le b_1(T-t) \\ 0 & \text{if} \quad b_1(T-t) < X_0(t) < b_2(T-t) \\ A-B & \text{if} \quad X_0(t) \ge b_2(T-t) \end{cases}$$

where

$$(2.16)\qquad dX_0(t) = u_0(X_0(t))dt + dW(t) + dN(t)$$

$$X_0(0) = x.$$

<u>Proof</u>. Let $D = V_{xx}$, (omitting (x,t) arguments).

<u>Claim</u>. $D \ge 0$ all (x,t).

<u>Proof of Claim</u>. From (2.4), omitting (x,t), let

$$(2.17) \qquad L(D) \equiv g(D_x) + \tfrac{1}{2}D_{xx} - (\alpha+\lambda)D - D_t = -\varphi_{xx} - \lambda D(x+1).$$

From (1.3), the conditions on φ, and (2.8) - (2.14), it follows that, for each t,

$$(2.18) \qquad D > 0 \text{ as } |x| \to \infty.$$

Suppose that there is an $r > b_2$ and a γ, $0 < \gamma < 1$ for fixed t, such that

(2.19) $D(x) < 0 \qquad b_2 < r-\gamma < x < r$

(2.19a) $D(r) = 0$

(2.20) $D(x) > 0, \quad x > r.$

We now obtain a contradiction to (2.19).

From (1.3), $\varphi_{xx} > 0$, hence (2.17) and (2.20) imply that

$$L(D) < 0, \qquad x \geq r-1.$$

It follows from a maximum principle (Lemma 1(after multiplying by -1)[1], p. 34) that D cannot have a negative minimum for the fixed t, for $x \geq r-\gamma$. From this and (2.7a), (2.18),(2.20), if follows that if D were negative for any $x \geq r-\gamma$, it would have a negative minimum, which is not allowed by the maximum principle. Hence $D \geq 0$ for $x \geq r-\gamma$, contradicting (2.19) and completing the claim for $x \geq b_2$.
For $x < b_2$, a similar argument using (2.7b), (2.18) yields that $D(x) > 0$ for $b_2-\delta < x < b_2$ for appropriate $0 < \delta < 1$. Continuing the argument by iteration yields that $D(x) \geq 0$ all x, for each t.

The claim implies that V_x is increasing in x for each t and hence that (2.5)-(2.14) indeed yields a solution to the Bellman equation (2.4).

To show u_0 is optimal, define, for $0 \leq t \leq T$

$$(2.21) \qquad K(X(t),t) \equiv V(X(t),T-t)e^{-\alpha t}.$$

Noting that $K(X(0),0) = V(x,T)$ and $K(X(T),T) = 0$, the Ito rule ([2], pp.125-126) applied to (2.21) for an admissible u and its corresponding $X(t)$ yields, upon integrating from 0 to T, and adding and subtracting appropriate terms, that

$$(2.22)\quad \int_0^T e^{-\alpha s}(\varphi(X(s)) + |u(X(s))|)ds - V(x,T) =$$

$$\int_0^T e^{-\alpha s}(\varphi(X(s)) + g(V_x(X(s),T-s)) + \tfrac{1}{2} V_{xx}(X(s),T-s)$$

$$-\alpha V(X(s),T-s) - V_t(X(s),T-s))ds$$

$$+ \int_0^T e^{-\alpha s} V(X(s),T-s)dN(s)$$

$$+ \int_0^T e^{-\alpha s} V_x(X(s),T-s)dW(s)$$

$$+ \int_0^T e^{-\alpha s}(u(X(s))V_x(X(s),T-s) + |u(X(s)| - g(V_x(X(s),T-s)))ds.$$

The fourth integral on the right side of (2.22) is non-negative. Upon taking expectations in (2.22), the third integral on the right becomes zero.

On combining the first and second integrals on the right after taking expectations, and suppressing the $X(s)$ and the $(X(s),T-s)$ arguments, one obtains from (2.22),

$$(2.23)\quad \int_0^T e^{-\alpha s} E(\varphi + |u|)ds - V(x,T) =$$

$$\int_0^T e^{-\alpha s} E(\varphi + g(V_x) + \tfrac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(X(s)+1,T-s) - V))ds$$

$$\int_0^T e^{-\alpha s} E(uV_x + |u| - g(V_x))ds.$$

The first integral on the right of (2.23) is zero by (2.4) and the second integral on the right is non-negative by definition of g in (2.3), with equality if $u=u_0$. Hence from (2.23),

$$(2.24) \quad \int_0^T e^{-\alpha s} E(\varphi + |u| ds \geq V(x,T)$$

with equality if $u=u_0$, showing that u_0 is optimal. This completes Theorem 1.

3. Infinite Interval Control. Assume that the conditions of section 2 hold and let $T = \infty$. The cost function is then

$$(3.1) \quad J(u) = \int_0^\infty e^{-\alpha s} E(\varphi(X(s)) + |u(X(s))|) ds$$

which is finite by (1.3) for admissible u.

Define $V \equiv V(x)$ as

$$(3.2) \quad V(x) = \inf_{|u-A| \leq B} \int_0^\infty e^{-\alpha s} E(\varphi(X(s)) + |u(X(s)|) ds$$

where $X(0) = x$, a constant.

By writing $\int_0^\infty \equiv \int_0^h + \int_h^\infty$, heuristic arguments (see[2], pp. 179-180), yield a Bellman equation, using the abbreviated arguments in (2.2),

$$(3.3) \quad \varphi(x) + \inf_{|u-A| \leq B} (uV_x + |u|) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1) - V) = 0.$$

As in (2.3), (2.4),

$$(3.4) \qquad \varphi(x) + g(V_x) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0$$

On heuristic grounds, a solution to (3.4) is sought such that for numbers $b_1 < b_2$ to be determined below, omitting x arguments,

$$(3.5a) \qquad \varphi + (A+B)(1+V_x) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0 \quad \text{for } x \le b_1$$

$$(3.5b) \qquad \varphi + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0 \quad \text{for } b_1 < x < b_2 ,$$

$$(3.5c) \qquad \varphi + (B-A)(1-V_x) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0 \quad \text{for } x \ge b_2.$$

The $b_1 < b_2$ are to be determined from the following matching conditions, where $V \equiv V_1$ in (3.5a), $V \equiv V_2$ in (3.5b) and $V \equiv V_3$ in (3.5c).

$$V_1(b_1) = V_2(b_1)$$

$$V_2(b_2) = V_3(b_2)$$

$$(3.6) \qquad V_{1,x}(b_1) = V_{2,x}(b_1) = -1$$

$$V_{2,x}(b_2) = V_{3,x}(b_2) = +1.$$

$$(3.6a) \qquad V_{3,xx}(b_2) \ge 0$$
$$V_{2,xx}(b_1) \ge 0.$$

For R a constant, denote

$$(3.7) \qquad J(x,R) \equiv \int_0^\infty e^{-\alpha s} E(\varphi(Rs+W(s)+N(s)+x)+|R|)ds.$$

It may be verified that $J(x,A+B)$ is a particular solution to (3.5a), that $J(x,0)$ is a particular solution to (3.5b) and $J(x,A-B)$ is a

particular solution to (3.5c).

Assumption 3. There is a non-zero solution $H_1(x)$ to (omitting x argument)

$$(3.8) \qquad (A+B)(1+H_x) + \frac{1}{2} H_{xx} - \alpha H + \lambda(H(x+1)-H) = 0$$

such that

$$(3.9) \qquad H_1(x) = O(e^{+ux})$$

$$H_{1,xx}(x) = O(e^{+vx})$$

for some $u > 0$, $v > 0$, as $x \to -\infty$.

There is a non-zero solution $H_2(x)$ to

$$(3.10) \qquad \frac{1}{2} H_{xx} - \alpha H + \lambda(H(x+1)-H) = 0.$$

There is a non-zero solution $H_3(x)$ to

$$(3.11) \qquad (B-A)(1-H_x) + \frac{1}{2} H_{xx} - \alpha H + \lambda(H(x+1)-H) = 0$$

such that

$$(3.12) \qquad H_3(x) = O(e^{-qx})$$

$$H_{3,xx}(x) = O(e^{-px})$$

for some $p > 0$, $q > 0$ as $x \to +\infty$.

Now one sets

$$(3.13a) \qquad V_1(x) = J(x,A+B) + H_1(x)$$

$$(3.13b) \qquad V_2(x) = J(x,0) + H_2(x)$$

$$(3.13c) \qquad V_3(x) = J(x,A-B) + H_3(x) .$$

Assumption 4.

(3.14) The $b_1 < b_2$ are determined by the $V_\ell(x)$, $1 \leq \ell \leq 3$ of (3.13 a-c).

Theorem 2. Under the assumptions of this section, the optimal $u = u_1$ may be expressed as

$$(3.15)\quad u_1(X_1(t)) = \begin{cases} A+B & \text{if } X_1(t) \leq b_1 \\ 0 & \text{if } b_1 < X_1(t) < b_2 \\ A-B & \text{if } X_1(t) \geq b_2 \end{cases}$$

where

$$(3.16)\quad dX_1(t) = u_1(X_1(t))dt + dW(t) + dN(t)$$
$$X_1(0) = x.$$

Proof. Let $D = V_{xx}$, suppressing the x-arguments.

Claim. $D \geq 0$ all x.

Proof of Claim. From (3.4)

$$(3.17)\quad K(D) \equiv g\,(D_x) + \frac{1}{2} D_{xx} - (\alpha+\lambda)D = -\varphi_{xx} - \lambda D(x+1).$$

By an argument identical to that given in the pooof of Theorem 1, using the appropriate maximum principle ([1], Theorem 18, p. 53), it follows that $D(x) \geq 0$ for all x.

The claim implies that V_x is increasing in x and hence that (3.5)-(3.14) yields a solution to (3.4).

To show u_1 is optimal, define, for $t \geq 0$,

$$(3.18)\qquad R(X(t)) \equiv V(X(t))e^{-\alpha t}.$$

Noting that $R(0) = V(x)$, the Ito rule ([2], pp. 125-126) applied to $R(X(t))$ followed by integration and adding and subtracting appropriate terms yields (arguments on the right side not indicated are $X(s)$)

$$\int_0^t e^{-\alpha s}(\varphi(X(s)) + |u(X(s))|)ds + V(X(t))e^{-\alpha t} - V(x) = \tag{3.19}$$

$$\int_0^t e^{-\alpha s}(\varphi + g(V_x) + \frac{1}{2} V_{xx} - \alpha V)ds + \int_0^t e^{-\alpha s} V dN(s)$$

$$+ \int_0^t e^{-\alpha s} V_x dW(s) + \int_0^t e^{-\alpha s}(\varphi + |u| - g(V_x))ds.$$

The fourth integral on the right is non-negative by the definition of $g(x)$. Upon taking expectations in (3.19), the third integral on the right vanishes, and the first and second terms on the right may be combined to obtain (again the arguments not indicated are $X(s)$)

$$\int_0^t e^{-\alpha s} E(\varphi + |u|)ds + e^{-\alpha t} EV(X(t)) - V(x) = \tag{3.20}$$

$$\int_0^t e^{-\alpha s} E(\varphi + g(V_x) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(X(s)+1) - V))ds$$

$$+ \int_0^t e^{-\alpha s} E(uV_x + |u| - g(V_x))ds.$$

By (3.4), the first integral on the right of (3.20) is zero, and the second integral on the right is non-negative by the definition of g. From (1.3), (3.5)-(3.14), and the bounds on $|u|$, it follows that there is a constant $L > 0$ such that for all t,

$$E(V(X(t)))e^{-\alpha t} \leq Le^{-\alpha t} \tag{3.21}$$

letting $t \to \infty$ in (3.20), and using (3.21) one obtains

$$(3.22) \qquad \int_0^\infty e^{-\alpha s} E(\varphi + |u|)ds \geq V(x)$$

and
$$(3.23) \qquad \int_0^\infty e^{-\alpha s} E(\varphi(X_1(s)) + |u_1(X_1(s))|)ds = V(x)$$

so that (3.22), (3.23) yield that u_1 is optimal, completing Theorem 2.

4. Alternate Cost Function. The same models as in sections 1-3 with certain other cost functions may be treated in a similar way. For example, the cost function (1.5) yields, for $T < \infty$, a Bellman equation (the (x,t) arguments are omitted)

$$(4.1) \qquad \varphi + h(V_x) + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t) - V) = 0$$

where

$$(4.2) \quad h(a) = \inf_{|u-A| \leq B} (ua + u^2) = \begin{cases} (A+B)(a+A+B) & \text{if } -\frac{a}{2} \geq A+B \\ -\frac{a^2}{4} & \text{if } A-B < -\frac{a}{2} < A+B \\ (A-B)(a+A-B) & \text{if } -\frac{a}{2} \leq A-B \end{cases}$$

and from (4.2),

$$(4.3) \quad u = A-B \text{ if } -\frac{a}{2} < A-B; \; u = -\frac{a}{2} \text{ if } A-B < -\frac{a}{2} < A+B;$$
$$u = A+B \text{ if } -\frac{a}{2} > A+B.$$

One seeks a solution to (4.1) of the form

$$(4.4a) \quad \varphi + (A+B)(V_x + A + B) + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1;t) - V) = 0$$
$$\text{if} \quad x \leq k_1(t)$$

$$(4.4b) \quad \varphi - \frac{1}{4}(V_x)^2 + \frac{1}{2} V_{xx} - \alpha V - V_t + \lambda(V(x+1,t) - V) = 0$$
$$\text{if} \quad k_1(t) < x < k_2(t)$$

and

$$\text{(4.4c)}\quad \varphi + (A-B)(V_x+A-B) + \frac{1}{2}V_{xx} - \alpha V - V_t + \lambda(V(x+1,t)-V) = 0$$

$$\text{if} \quad x \geq k_2(t).$$

The $k_1(t)$, $k_2(t)$ are to be determined by, for $V \equiv V_1$ in (4.4a), $V \equiv V_2$ in (4.4b), $V \equiv V_3$ in (4.4c), for $0 \leq t \leq T$,

$$\text{(4.5)}\quad \begin{aligned} V_1(k_1(t),t) &= V_2(k_1(t),t) \\ V_2(k_2(t),t) &= V_3(k_2(t),t) \\ V_{1,x}(k_1(t),t) &= V_{2,x}(k_1(t),t) = -2(A+B) \\ V_{2,x}(k_2(t),t) &= V_{3,x}(k_2(t),t) = -2(A-B). \end{aligned}$$

and

$$\text{(4.6)}\quad V_1(x,0) = V_2(x,0) = V_3(x,0) = 0 \text{ all } x.$$

$$\text{(4.6a)}\quad \begin{aligned} &V_{2,xx}(k_1(t),t) \geq 0 \\ &V_{3,xx}(k_2(t),t) \geq 0. \end{aligned}$$

The maximum principle ([1], Lemma 1 p. 34) may be applied to (4.4a), (4.4c) as in Theorem 1, as these are linear in the V_x term. Similarly, for R a constant, if one defines

$$\text{(4.7)}\quad L(x,t,R) \equiv \int_0^t e^{-\alpha s} E(\varphi(Rs+W(s)+N(s)+x)+R^2)ds ,$$

then $L(x,A+B)$ is a particular solution to (4.4a) and $L(x,A-B)$ is a particular solution to (4.4c). Adding an assumption similar to

assumptions 1 and 2 in section 2, and on the boundedness of solutions to (4.4b), one may obtain the optimal u_3 for this problem implicitly in the form

$$(4.8)\quad u_3(X_3(t),t) = \begin{cases} A+B & \text{if} \quad X_3(t) \le k_1(T-t) \\ -\frac{1}{2}V_x(X_3(t),t) & \text{if} \quad k_1(T-t) < X_3(t) < k_2(T-t) \\ A-B & \text{if} \quad X_3(t) \ge k_2(T-t), \end{cases}$$

where

$$dX_3(t) = u_3(X_3(t))dt + dN(t) + dW(t),$$

using arguments as in Theorem 1.

The case $T = \infty$ parallels that of section 3. The Bellman equation is, for $V \equiv V(x)$,

$$(4.9)\quad \varphi + h(V_x) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0$$

and a solution to (4.9) is sought of the form

$$(4.10a)\quad \varphi + (A+B)(V_x + A+B) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0$$

$$\text{for} \quad x \le \ell_1$$

$$(4.10b)\quad \varphi - \frac{1}{4}(V_x{}^2) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0$$

$$\text{for} \quad \ell_1 < x < \ell_2 ,$$

and

$$(4.10c)\quad \varphi + (A-B)(V_x + A-B) + \frac{1}{2} V_{xx} - \alpha V + \lambda(V(x+1)-V) = 0$$

$$\text{for} \quad x \ge \ell_2$$

where the constants $\ell_1 < \ell_2$ are to be determined from the matching conditions where $V \equiv V_1$ in (4.10a), $V \equiv V_2$ in (4.10b), $V \equiv V_3$ in (4.10c)

$$V_1(\ell_1) = V_2(\ell_1)$$

$$V_2(\ell_2) = V_3(\ell_2)$$

(4.12)

$$V_{1,x}(\ell_1) = V_{2,x}(\ell_1) = -2(A+B)$$

$$V_{2,x}(\ell_2) = V_{3,x}(\ell_2) = -2(A-B).$$

(4.11a) $$V_{2,xx}(\ell_1) \geq 0$$

$$V_{3,xx}(\ell_2) \geq 0.$$

If

$$(4.12) \quad L(x,R) \equiv \int_0^\infty e^{-\alpha s}(E(\varphi(Rs+W(s)+N(s)+x)+R^2)ds,$$

it may be shown that $L(x,A+B)$ is a particular solution to (4.10a) and $L(x,A-B)$ is a particular solution to (4.10c). Adding an appropriate assumption similar to that in section 3, and on the boundedness of solutions to (4.10b), the optimal u_4 is implicitly expressed as

$$(4.13) \quad u_4(X_4(t)) = \begin{cases} A+B & \text{if} \quad X_4(t) \leq \ell_1 \\ -\frac{1}{2}V_x(X_4(t)) & \text{if} \quad \ell_1 < X_4(t) < \ell_2 \\ A-B & \text{if} \quad X_4(t) \geq \ell_2 \end{cases}$$

where

$$dX_4(t) = u_4(X_4(t))dt + dW(t) + dN(t)$$

$$X_4(0) = x .$$

5. Additional Constraints.

Certain additional constraints may be incorporated and treated by those methods. For example, in the case $T < \infty$ of section 2, the added constraint

$$(5.1)\quad E(\varphi(X(a)) + |u(X(a))|) = C,$$

where a is a constant, $0 < a \le T$ and $C > 0$, may be incorporated by adding the condition

$$V_t'(x,t)\Big|_{t=a} = e^{-\alpha a}C$$

to the conditions (2.6),(2.7), and proceeding as before. See [3] for another approach.

6. Extensions.

The method applies to a variant of the stochastic differential equation (1.1). Let, for $\beta \neq 0$ a constant,

$$(6.1)\quad X_t(t) = (\beta X(t)+u(X(t))dt + dW(t)+dN(t),$$

$$X(0) = x,$$

with control $u(X(t))$ satisfying (1.2) as before. Similarly, the cost function $J(u)$ is as in (1.4).

The appropriate Bellman equation for $T < \infty$ is, where g is as in (2.3),

$$(6.2)\qquad 0 = \varphi(x)+xV_x+g(V_x)+ \frac{1}{2} V_{xx}-V_t+\lambda(V(x+1)-V)-\alpha V.$$

A solution of the form (2.5)-(2.7) is sought as before. To obtain a particular solution to, e.g., the Bellman equation

$$(6.3)\qquad 0 = \varphi(x)+xV_x+(A+B)(1+V_t)+ \frac{1}{2}V_{xx}-\alpha V-V_t$$

$$+\lambda(V(x+1,t)-V) = 0,$$

denote

$$(6.4) \quad Y(t) = e^{-\beta t} X(t).$$

Then

$$(6.5) \quad dY(t) = e^{-\beta t}(A+B)dt + e^{-\beta t} dW(t) + e^{-\beta t} dN(t)$$

Hence, integrating (6.5)

$$(6.6) \quad Y(t) = x + \int_0^t e^{-\beta s} dW(s) + \int_0^t e^{-\beta s} dN(s) + (A+B) \int_0^t e^{-\beta s} ds$$

or

$$(6.7) \quad X(t) = xe^{\beta t} + \frac{(A+B)}{\beta}(e^{\beta t}-1) + \int_0^t e^{\beta(t-s)} dW(s) + \int_0^t e^{\beta(t-s)} dN(s).$$

It may now be verified that

$$(6.8) \quad J(x,t,A+B) \equiv E \int_0^t (\varphi(xe^{\beta t} + \frac{(A+B)}{\beta}(e^{\beta t}-1) + \int_0^t e^{\beta(t-s)} dW(s) + \int_0^t e^{\beta(t-s)} dN(s)) + (A+B))ds$$

is a particular solution to (6.3). The rest of the construction and matching and initial conditions and proofs are as in section 2.

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A one-dimensional Wiener plus independent Poisson control process has integrated, discounted non-quadratic cost function with asymmetric bounds on the non-anticipative control, assumed to be a function of the current state. A Bellman equation and maximum principle for partial differential-difference equations may be used to obtain the optimal closed loop control if some assumptions on the asymptotic behavior of certain partial differential-difference equations are met. The finite and infinite integral cases are treated separately.

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